JOURNAL OF APPROXIMATION THEORY **86**, 270–277 (1996)
ARTICLE NO. 0068

One Example in Monotone Approximation

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Communicated by Dany Leviatan

Received December 19, 1994; accepted in revised form September 4, 1995

For each non-negative integer n a function $f=f_n$ is constructed such that f has a continuous and non-negative derivative f' on $I := [-1, 1]$ and

$$\frac{1}{200} < \frac{E^{(1)}_{n+1}(f)}{E_n(f')} < 2,$$

where $E_n(f')$ ($E^{(1)}_{n+1}(f)$) is the value of the best uniform approximation on I of the function f' (f) by arbitrary (monotone on I) algebraic polynomials of degree $\leqslant n$ ($n+1$).

INTRODUCTION

Let n be a non-negative integer and $\mathcal{P}_n$ the space of all algebraic polynomials of degree at most n. For a function f, continuous on a closed interval $[a, b]$, set, as usual,

$$\|f\|_{[a,b]} := \max_{x \in [a,b]} |f(x)|,$$

and denote

$$\|f\| := \|f\|_{[-1,1]}, \qquad E_n(f) := \inf_{p \in \mathcal{P}_n} \|f - p\|, \qquad I := [-1, 1].$$

If a function f is continuously differentiable on I, then it is well known that

$$E_{n+1}(f) \leqslant \frac{c}{n+1} E_n(f'), \tag{1}$$

where c is an absolute constant.

* Supported by the International Science Foundation (Soros) and the State Fund for Fundamental Researches of Ukraine.

0021-9045/96 $18.00

Now, let $\Delta^{(1)}$ be the set of functions f that have continuous and non-negative derivatives f' on I, and

$$E_n^{(1)}(f) := \inf_{p \in \mathcal{P}_n \cap \Delta^{(1)}} \|f - p\|.$$

D. Leviatan noted that Shisha's paper [1] contains a proof of the inequality

$$E_{n+1}^{(1)}(f) \leqslant 2E_n(f') \tag{2}$$

for each function $f \in \Delta^{(1)}$. Let us review it. Denote by $P_n^*(x, f') =: P_n^* \in \mathcal{P}_n$ the polynomial of the best approximation of f'. Since $f \in \Delta^{(1)}$, then $P_n^*(x, f') + E_n(f') \geqslant 0$ for $x \in I$. Let

$$P_{n+1}(x, f) := f(0) + xE_n(f') + \int_0^x P_n^*(u, f')\, du,$$

then, since $P_{n+1} \in \Delta^{(1)}$, we have

$$E_n^{(1)}(f) \leqslant \|f - P_{n+1}\| \leqslant E_n(f') + \|f' - P_n^*(\cdot, f')\| = 2E_n(f').$$

Taking into account the works on monotone approximation of Lorentz, DeVore, and many other authors, one may assume that the estimate (2) can be strengthened, say, to the form (1), where $E_{n+1}(f)$ is replaced by $E_{n+1}^{(1)}(f)$. In fact, S. V. Koniagin and A. S. Shvedov asked the author this question in 1989 in Lutsk, during the conference dedicated to Professor V. K. Dzjadyk's 70th anniversary.

In this paper the following Theorem 1 will be proved which gives a negative answer to this question. It turns out that the estimate (2) is exact in order.

THEOREM 1. *For each non-negative integer n there is a function $f = f_n$, which has a continuous and non-negative on $[-1, 1]$ derivative f', such that*

$$\frac{1}{200} < \frac{E_{n+1}^{(1)}(f)}{E_n(f')} < 2. \tag{3}$$

Everywhere below $n > 2$ is a fixed positive integer.

1. *A Few Simple Properties of Chebyshev Polynomials.* Let us consider Chebyshev polynomial of degree n

$$T_n(x) := \cos n \arccos x, \qquad \text{if} \quad |x| \leqslant 1, \tag{4}$$

$$T_n(x) := \tfrac{1}{2}((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n), \qquad \text{if} \quad |x| \geqslant 1. \tag{5}$$

It follows from (5) that

$$|T_n(x)| < \tfrac{1}{2}(2\,|x|)^n, \qquad |x| > 1. \tag{6}$$

For an arbitrary polynomial $S_n \in \mathcal{P}_n$ the following inequality (see [3, 1.2.9]) is well known:

$$|S_n(x)| \leqslant |T_n(x)|\,\|S_n\|, \qquad |x| > 1. \tag{7}$$

Hence, by taking (6) into account, we get

$$\|S_n\|_{[-3,\,1]} < \tfrac{1}{2}6^n\,\|S_n\|,$$

and, in particular,

$$\|S_n\| < \tfrac{1}{2}6^n\,\|S_n\|_{[0,\,1]}. \tag{8}$$

Denote by

$$x_k := x_{k,\,n} := \cos\left(\frac{n-k}{n}\pi + \frac{\pi}{2n}\right), \qquad k = \overline{1, n},$$

the zeroes of Chebyshev polynomial T_n. For $k = \overline{1, n-1}$ set $I_k := [x_k, x_{k+1}]$, $|I_k| := x_{k+1} - x_k$. The formula of the area of a triangle gives

$$\int_{I_k} |T_n(u)|\,du > \tfrac{1}{2}|I_k|,$$

for all $k = \overline{1, n-1}$; a more accurate calculation gives

$$\int_{I_k} |T_n(u)|\,du = \frac{n}{n^2-1}\operatorname{ctg}\frac{\pi}{2n}\,|I_k| > \frac{2}{\pi}\,|I_k|. \tag{9}$$

Clearly,

$$|I_k| = |I_{n-k}|, \qquad |I_k| \leqslant |I_{k+1}| \tag{10}$$

for all $k = \overline{1, [(n-1)/2]}$, where $[\,\cdot\,]$ stands for the integer part.

We also put

$$Q_{n+1}(x) := \int_{-1}^{x} T_n(u)\,du \tag{11}$$

and note that

$$\|Q_{n+1}\| \leqslant \frac{1}{n-1}.$$

2. *Definition of a Function f.* Let

$$b := \frac{1}{14}; \tag{12}$$

$$I^* := [-1, -1+2b]; \qquad T_n^*(x) := T_n\left(\frac{x+1-b}{b}\right); \tag{13}$$

and let $x_k^* := -1 + b + bx_k$ denote zeroes of the polynomial $T_n^*(x)$; $I_k^* := [x_k^*, x_{k+1}^*]$.

Now, we define the function f by the formula

$$f(x) := f_n(x) := \begin{cases} 0, & \text{if } \quad x \in [-1, x_n^*], \\ \int_{x_n^*}^{x} T_n^*(u)\, du, & \text{if } \quad x \in [x_n^*, 1]. \end{cases} \tag{14}$$

Obviously,

$$f \in \Delta^{(1)}. \tag{15}$$

3. *Beginning of the Proof of Theorem* 1. First of all, note that

$$\begin{aligned} E_n(f') &\leqslant \| f' - T_n^* \| = \| f' - T_n^* \|_{[-1, x_n^*]} = \| T_n^* \|_{[-1, x_n^*]} \\ &\leqslant \| T_n^* \|_{I^*} = \| T_n \| = 1. \end{aligned} \tag{16}$$

Taking into account (2), (15), and (16), we only have to prove the inequality

$$E_{n+1}^{(1)}(f) > \tfrac{1}{200}.$$

For this purpose let us fix an arbitrary polynomial $P_{n+1} \in \mathscr{P}_{n+1}$ such that

$$P'_{n+1}(x) \geqslant 0, \tag{17}$$

for all $x \in I$, and prove the estimate

$$\| f - P_{n+1} \| > \tfrac{1}{200}. \tag{18}$$

Set

$$Q_{n+1}^*(x) := \int_{-1}^{x} T_n^*(u)\, du, \qquad R_{n+1}(x) := P_{n+1}(x) - Q_{n+1}^*(x).$$

Denote by m the number of all zeroes (counting their multiplicities) of the polynomial R'_{n+1}, which lie in the interval $\langle x_1^*, x_n^*)$ where $\langle$ stands for (when n is even and for [when n is odd. We shall write $m=0$ if R'_{n+1} has no zeroes in $\langle x_1^*, x_n^*)$. Clearly, $m \leqslant n$.

In the next sections we shall prove (18) for two cases, $m < n/2$ (including $m=0$) and $m \geqslant n/2$.

4. *Proof of the Estimate* (18) *in the Case* $m < n/2$. As usual, for a finite collection V of some elements we denote by card V the number of elements in V.

We shall write $k \in W$ if $n-k$ is even and $1 \leqslant k \leqslant n-1$, that is, if $T_n^*(x) \geqslant 0$ for $x \in I_k^*$ (or, which is the same, if $T_n(x) \geqslant 0$ for $x \in I_k$). Note that card $W = [(n-1)/2]$. We shall write $k \in W_0$, if $k \in W$ and the polinomial R'_{n+1} has a zero on I_k^*. Note that when $k \in W_0$, the polynomial R'_{n+1} must have an even number of zeros (counting their multiplicities) on I_k^*, because $R'_{n+1} > 0$ for $x \in (x_{k-1}^*, x_k^*) \cup (x_{k+1}^*, x_{k+2}^*)$. Therefore card $W_0 \leqslant m/2$, and

$$\left[\frac{n-1}{2}\right] \geqslant l := \operatorname{card}(W \setminus W_0) \geqslant \left[\frac{n-1}{2}\right] - \frac{m}{2}. \tag{19}$$

It follows from (17) and our construction that, for $x \in I_k^*$ with $k \in W \setminus W_0$,

$$P'_{n+1}(x) > T_n^*(x) \geqslant 0. \tag{20}$$

By applying (17), (20), and (13) we get

$$\begin{aligned}\int_{x_1^*}^{x_n^*} P'_{n+1}(x)\,dx &\geqslant \sum_{k \in W \setminus W_0} \int_{I_k^*} P'_{n+1}(x)\,dx \\ &> \sum_{k \in W \setminus W_0} \int_{I_k^*} T_n^*(x)\,dx = b \sum_{k \in W \setminus W_0} \int_{I_k} T_n(x)\,dx \\ &= b \sum_{k \in W \setminus W_0} \int_{I_k} |T_n(x)|\,dx =: bA.\end{aligned}$$

Now, taking into account that if $k \in W$ then $(k \pm 1) \notin W$, and using (9), (10), and (19), we find

$$A > \frac{2}{\pi} \sum_{k \in W \setminus W_0} |I_k| \geqslant \frac{2}{\pi} \sum_{k=1}^{l} |I_k| = \frac{2}{\pi}(x_{l+1} - x_1).$$

Finally, we write a chain of obvious relations,

$$\begin{aligned}\|f-P_{n+1}\| &\geqslant \|f-P_{n+1}\|_{[x_1^*, x_n^*]} = \|P_{n+1}\|_{[x_1^*, x_n^*]}\\ &\geqslant \frac{1}{2}(P_{n+1}(x_n^*) - P_{n+1}(x_1^*))\\ &= \frac{1}{2}\int_{x_1^*}^{x_n^*} P'_{n+1}(x)\,dx > \frac{b}{2}A\\ &> \frac{b}{\pi}(x_{l+1} - x_1)\\ &= \frac{2b}{\pi}\sin\frac{l+1}{2n}\pi\,\sin\frac{l}{2n}\pi.\end{aligned}$$

Note that $l \geqslant [(n+1)/4]$, and (when $n \neq 6$) we obtain

$$\|f-P_{n+1}\| > \frac{2b}{\pi}\sin\frac{\pi}{8}\sin\left(\left[\frac{n+1}{4}\right]\frac{\pi}{2n}\right) \geqslant \frac{2b}{\pi}\sin\frac{\pi}{8}\sin\frac{\pi}{10} > \frac{1}{200}.$$

The estimate (18) is proved in the case $m < n/2$.

Remark. Then proof in the following section was derived by Professor S. V. Koniagin. The author's original proof was more complicated as it made use of V. A. Markov's inequality, whereas here we make use of inequality (7).

5. *Proof of the Estimate* (18) *in the Case* $m \geqslant n/2$. Let t_i, $i = \overline{1, m}$, denote those zeroes of the polynomial R'_{n+1} which lie in the interval $\langle x_1^*, x_n^*)$, and let S_{n-m} be the polynomial $S_{n-m} \in \mathscr{P}_{n-m}$ which, for $x \neq t_i$, is defined by the formula

$$S_{n-m}(x) := \frac{R'_{n+1}(x)}{\Pi(x)},$$

where

$$\Pi(x) := \prod_{i=1}^{m} (x - t_i).$$

Also, by $a := -1 + b + b\cos(\pi/n)$ we denote the right-hand minimum of the polynomial T_n^*. Since $T_n^*(a) = -1$, $P'_{n+1}(a) \geqslant 0$, then

$$S_{n-m}(a) \geqslant \Pi^{-1}(a)$$

and, therefore, by (8)

$$\|S_{n-m}\|_{[0,1]} > 2 \cdot 6^{m-n} \|S_{n-m}\| \geqslant 2 \cdot 6^{m-n} \Pi^{-1}(a).$$

Now,

$$\frac{\Pi(0)}{\Pi(a)} > (a+1)^{-m} > 7^m, \qquad \|R'_{n+1}\|_{[0,1]} \geqslant \Pi(0) \|S_{n-m}\|_{[0,1]},$$

and hence

$$\begin{aligned} \|R'_{n+1}\| \geqslant \|R'_{n+1}\|_{[0,1]} &\geqslant \Pi(0) \|S_{n-m}\|_{[0,1]}, \\ &\geqslant 6^{m-n} 2\Pi(0)\, \Pi^{-1}(a) \geqslant 6^{m-n} 7^m 2 \geqslant 2(7/6)^{n/2}. \end{aligned} \tag{21}$$

Now let us make use of A. A. Markov's inequality

$$\|R_{n+1}\| \geqslant \frac{1}{(n+1)^2} \|R'_{n+1}\| \tag{22}$$

and of arguments used by G. G. Lorentz and K. L. Zeller [2] and A. S. Shvedov [4].

Obviously,

$$\begin{aligned} \|f - Q^*_{n+1}\| &= \|f - Q^*_{n+1}\|_{[-1, x^*_n]} = \|Q^*_{n+1}\|_{[-1, x^*_n]} \\ &= b\, \|Q_{n+1}\|_{[-1, x_n]} \leqslant b\, \|Q_{n+1}\|, \end{aligned}$$

and (11) yields

$$\|f - Q^*_{n+1}\| \leqslant \frac{b}{n-1}. \tag{23}$$

Finally, the estimates (21)–(23) and (12) imply

$$\begin{aligned} \|f - P_{n+1}\| \geqslant \|R_{n+1}\| - \|f - Q^*_{n+1}\| &\geqslant \frac{1}{(n+1)^2} \|R'_{n+1}\| - \frac{b}{n-1} \\ &\geqslant \frac{2(7/6)^{n/2}}{(n+1)^2} - \frac{1}{14(n-1)} > \frac{1}{200}, \end{aligned}$$

which completes the proof of the inequality (18) in the case $m \geqslant n/2$.

Theorem 1 is proved.

Remark. One can easily verify the cases $n = 0, 1, 2$ of Theorem 1. Also, we mention that we can double the constant 1/200 in (3), but for this purpose we would need to double the volume of the article.

Acknowledgments

The author is very indebted to Professors S. V. Koniagin and D. Leviatan for useful discussions on this article.

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